Finite-temperature structure factor in the Haldane–Shastry spin chain

2003 J. Phys. A: Math. Gen. 36 7233

(http://iopscience.iop.org/0305-4470/36/26/302)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 7233–7249 PII: S0305-4470(03)59790-6

Finite-temperature structure factor in the Haldane–Shastry spin chain

Stéphane Peysson

Institute for Theoretical Physics, University of Amsterdam, Vlackenierstraat 65, 1018 XE Amsterdam, The Netherlands

Received 18 February 2003, in final form 15 May 2003
Published 18 June 2003
Online at stacks.iop.org/JPhysA/36/7233

Abstract

The Haldane–Shastry spin chain can be mapped to the infinite coupling limit of the $SU(2)$ spin Calogero–Sutherland model. We use the $\mathfrak{gl}_2$ Jack polynomial technology to compute the form factors of the spin operator on the multi-spinon spectrum. The spin structure factor is obtained through a form factor expansion. The expansion is proved to converge in the small momentum limit. Numerics based on two- and four-spinon contributions give an approximate result for the infinite temperature static and dynamic spin structure factor.

PACS number: 75.10.Pq

1. Introduction

Low-dimensional systems constitute fertile breeding grounds for exotic types of physical excitations. Fractionalization of quantum numbers such as charge and spin is known to take place, respectively, in one-dimensional interacting electron liquids and spin chains: in cases such as these, one must forget about weakly coupled particles, and instead adopt a whole new starting point for the description of the strongly coupled physics. Obviously, the identification and proper description of this new starting point is often a very involved and risk-prone process.

In this respect, quasi-one-dimensional spin systems have provided one of the sturdiest arenas. Experimental realizations of systems with fractionalized excitations are numerous and well documented. Probably the clearest and best-studied signature comes from neutron scattering experiments on effectively one-dimensional antiferromagnetic spin-$1/2$ chains [1]. The excitations seen are not the naively expected spin-1 spin waves, but rather gapless 'spinons', which one could loosely present as spin-$1/2$ spin waves. Among many remarkable properties of these excitations are their fractional statistics, intermediate between fermions and bosons, making such a system markedly different from the one obeying convential rules.

On the theoretical side, strongly-coupled systems such as spin chains have in the last few decades presented extreme, if not seemingly insurmountable, difficulties. The simplest way of explaining this fact might be to say that quantum fluctuations are very strong in one

dimension, and cannot be tamed by perturbative approaches. Instead, excitations are strongly nonlinear, and one is faced with the seemingly impossible challenge of either providing an exact solution or risking missing out completely on the correct physics.

The quantity of interest to experimentalists (thinking about neutron scattering experiments) is the dynamical spin structure factor (DSSF). For a chain of N spins at sites R_i, this is defined as

$$S^{\alpha\beta}(\mathbf{q},\omega;T)=\frac{1}{2\pi N}\sum_{i,j}\mathrm{e}^{\mathrm{i}\,\mathbf{q}\cdot(\mathbf{R}_j-\mathbf{R}_i)}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\omega t}\left\langle S_i^{\alpha}(0)S_j^{\beta}(t)\right\rangle_T \tag{1}$$

where the angular brackets denote a thermal average.

The model of choice for the description of the spin dynamics depends of course on the specifics of the experimental setup one wishes to describe. The XXZ Heisenberg model [2] often fits the bill remarkably well, at least for very low energies. The Bethe ansatz method [3] could provide most of its thermodynamic properties, but little about its dynamics. Approximate methods have thus been used to address the computation of the DSSF. The Müller ansatz [4] is the best conjecture for the zero-temperature structure factor based on exact results and numerics. Finite-temperature low-energy features were obtained by Schultz using bosonization [5].

To go beyond the field theory limit requires tackling the nonlinear nature of the original model. Considerable insight in this direction was provided by the algebraic Bethe ansatz method [6] and the quantum inverse scattering theory [7]. Bougourzi *et al* used results from the algebraic analysis to compute the exact two-spinon contribution to the DSSF of the one-dimensional Heisenberg model [8–10]. More recently, Maillet *et al* proved multiple integral representations of elementary blocks of the correlation functions [11]. Nevertheless, all these approaches restrict to zero temperature and no exact thermodynamic limit is known. The computation of the DSSF at finite temperature requires a lot of further effort.

However, one of the properties of the Heisenberg model is that the spinons, though deconfined, still suffer from a residual interaction. The spinons are thus not truly free excitations obeying fractional statistics. There exists, on the other hand, a very convenient alternative approach based on the Haldane–Shastry model, whose Hamiltonian is [12, 13]

$$H_{\mathrm{HS}}=J\sum_{i<j}\frac{1}{[d(i-j)]^2}\mathbf{S}_i\cdot\mathbf{S}_j \tag{2}$$

where $d(i)=\frac{N}{\pi}\sin\frac{\pi i}{N}$. It is in the same universality class as the Heisenberg model, and the long-distance (decaying as $1/R^2$ for large distances) interaction in fact simplifies things considerably: the spinons form an ideal gas [14] of particles obeying fractional exclusion statistics [15]. The DSSF at $T=0$ can in fact be calculated exactly, and is given in the thermodynamic limit by [16]

$$\begin{aligned}
&S^{\alpha\beta}_{\mathrm{HS}}(\mathbf{q},\omega)=\frac{\delta_{\alpha\beta}}{2}\frac{\Theta(\omega_2(q_\parallel)-\omega)\Theta(\omega-\omega_{1-}(q_\parallel))\Theta(\omega-\omega_{1+}(q_\parallel))}{\sqrt{(\omega-\omega_{1-}(q_\parallel))(\omega-\omega_{1+}(q_\parallel))}}\\
&\omega_{1-}(q_\parallel)=\frac{J}{2}q_\parallel(\pi-q_\parallel)\qquad \omega_{1+}(q_\parallel)=\frac{J}{2}(q_\parallel-\pi)(2\pi-q_\parallel)\\
&\omega_2(q_\parallel)=\frac{J}{4}q_\parallel(2\pi-q_\parallel).
\end{aligned} \tag{3}$$

This $T=0$ formula is made up only of contributions from the two-spinon channel: all higher channels have vanishing contributions in the zero-temperature limit.

One of the very nice features of the Haldane–Shastry model is that its dynamics turn out to be much more easily tractable than those of the Heisenberg chain. The Haldane–Shastry model is but the first representative in a wider class of solvable models dubbed the

$SU(N)$ Haldane–Shastry chains. These are in turn obtainable as a particular limit of spin Calogero–Sutherland models, for which an impressive number of exact results are known in the mathematical literature. In particular, there exists a Yangian symmetry leading to the identification of a set of eigenvectors constructed from $\mathfrak{gl}_N$ Jack polynomials [17]. In short, the whole set of expectation values and transition matrix elements one might want to calculate in the Haldane–Shastry model turn out to have a correspondence in terms of Jack polynomials. More details on this will be provided in the bulk of the paper.

Thus, this opens the way to the computation of the DSSF (1) at nonzero temperatures for the Haldane–Shastry model. Our strategy will be to make use of the technology contained in [17] to compute the form factors involved in the spin–spin correlation function needed for the DSSF. This approach has already been used for $T = 0$ dynamical properties of the Haldane–Shastry spin chain [18, 19] and related models (spin Calogero–Sutherland model [20], supersymmetric t–J model [21, 22]). The present work is the first to address finite-temperature dynamics.

However, for a generic quantum field theory divergences appear when developing a correlation function on the Hilbert space. In the context of integrable field theories (ITF), it has been proposed that it could be rewritten as a sum free of divergences. The resulting formula, called a 'form factor expansion' (FFE), can be evaluated using the scattering data of the ITF. There is still an ongoing discussion about how precisely the FFE can be implemented in ITF [23–28]. For the case of conformal field theory (CFT), there has been a similar but independent proposal for writing finite-temperature correlators in a FFE [29]. It is based on the fractional statistics of the quasi-particles building the Hilbert space. Evidence was put forward that it converges quickly to the exact (known) result in terms of the number of excited quasi-particles [29, 30]. We will prove in the paper that the latter approach applies in the case of the Haldane–Shastry spin chain.

Our paper is organized as follows. First, we recall all the necessary aspects of the computation of form factors for the Haldane–Shastry model using Jack polynomials. We then set out to calculate the form factors themselves, in increasing complexity of spinon channels. The form factor expansion is then introduced and proved. We finally put the results together to provide an expression for the finite-temperature static and dynamical spin structure factors. Discussions and conclusions are amassed at the end.

2. Jack polynomial technology for the spin Calogero–Sutherland model

Using the freezing trick [31] the $SU(2)$ Haldane–Shastry model is obtained by the strong coupling limit of the $SU(2)$ spin Calogero–Sutherland model [32]. The latter describes N particles with coordinates $\{x_i, i = 1, \ldots, N\}$ moving on a circle of length N with the Hamiltonian

$$H_{\text{spin CS}} = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \frac{\pi^2}{2N^2}\sum_{i\leqslant j}\frac{\beta(\beta + P_{ij})}{\sin^2\left(\frac{\pi}{N}(x_i - x_j)\right)} \tag{4}$$

where P_{ij} is the $SU(2)$ exchange between particles i and j. Within the freezing trick, the interaction parameter β is taken to infinity. The particles are therefore pinned at $x_i = i$ and interact via the $SU(2)$ spin exchange.

The spin Calogero–Sutherland model is more tractable than the Haldane–Shastry model since it is continuous and all its eigenfunctions have been explicitly constructed. We recall in the following the Uglov construction [17] in terms of $\mathfrak{gl}_2$ Jack polynomials. The mathematical technology they provide to compute transition matrix elements is then introduced.

2.1. Yangian Gelfand–Zetlin basis

We consider the case N even and $N/2$ odd, so that the ground state is unique. Uglov determined the so-called Yangian Gelfand–Zetlin basis of this model, orthogonal through the Yangian action. They are labelled by the strictly decreasing sequences $k = \{k_i, i = 1, \ldots, N\}$, $k^0 = \{N/2+2-i, i = 1, \ldots, N\}$ corresponding to the ground state. k contains information on both momentum and spin of the excitation. Writing $k_i = 2\overline{k_i} + \underline{k_i}$, $\overline{k_i} \in \mathbb{Z}$ represents momentum and $\underline{k_i} \in \{1, 2\}$ colour[1]. More precisely, the momentum, the energy and the spin of a state described by k are

$$P_k = \frac{2\pi}{N} \sum_i \overline{k_i} \tag{5}$$

$$E_k = \frac{2\pi^2}{N^2} \sum_i [\overline{k_i} + \beta(N + 1 - 2\mathrm{i})/2]^2 \tag{6}$$

$$S_k = \frac{1}{2} \sum_i \left[\delta_{\underline{k_i},2} - \delta_{\underline{k_i},1}\right]. \tag{7}$$

For physical applications it is more convenient to work with excitations over the ground state. One identifies the state k with the pair

$$k \equiv \left(\lambda = (k_i - k_N + i - N, i = 1, \ldots, N-1), r = k_N - k_N^0\right). \tag{8}$$

It consists of a zero mode r and a partition λ (nonincreasing sequence of positive integers) of length $N-1$. The Hilbert space is spanned by all possible pairs. We refer the reader to [17] for the expressions of the physical properties in terms of (λ, r). They will be specified in the next section for the specific case of the Haldane–Shastry spin chain.

2.2. Uglov's isomorphism

Uglov determined an isomorphism Ω between the Yangian Gelfand–Zetlin basis and the $\mathfrak{gl}_2$ Jack polynomials defined through

$$\Omega(k) = (x_1 \ldots x_n)^r P_\lambda^{(2\beta+1,2)}(\{x_i\}) \tag{9}$$

where the Jack polynomial $P_\lambda^{(\gamma,2)}$ is the limit $q = -p$, $t = -p^\gamma$, $p \to 1$ of the Macdonald polynomial $P_\lambda(q, t)$. Then one has

$$(k, l)_{(\beta,2)} = \langle \Omega(k), \Omega(l) \rangle_{(\beta,2)}. \tag{10}$$

$(\cdot\,,\cdot)_{(\beta,2)}$ is the Yangian scalar product (see [17] for a definition), and $\langle \cdot\,,\cdot \rangle_{\beta,2}$ is the following scalar product in the space of symmetric Laurent polynomials:

$$\langle f(\{x_i\}), g(\{x_i\}) \rangle_{\beta,2} = \frac{1}{N!} \prod_{j=1}^{N} \int \frac{\mathrm{d}x_j}{2\mathrm{i}\pi x_j} \overline{f(\{x_i\})} \times \left[\prod_{1 \leqslant k \neq l \leqslant N} \left(1 - x_k^2 x_l^{-2}\right)^\beta \left(1 - x_k x_l^{-1}\right) \right] g(\{x_i\}). \tag{11}$$

The physical quantity we study in this work is the action of the spin operator. Using Uglov's isomorphism, it is given by (we dropped the scalar product indices for convenience)

[1] We use here a notation different from that of Uglov. In fact, this corresponds to the dual representation, called $*$ in his paper. Results are not altered, as one has to sum both representations to obtain physical quantities. Our choice for this representation is mostly practical, leading to states with positive momentum when σ_i is positive.

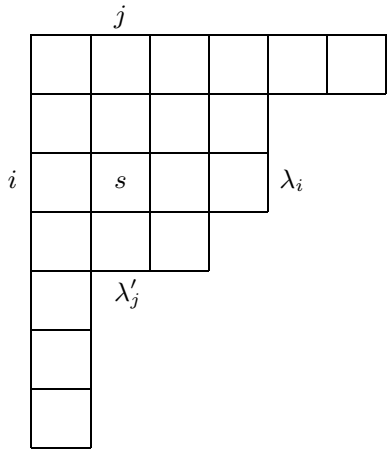

Figure 1. Partition $\lambda = (6, 4, 4, 3, 1, 1, 1)$.

$$(\lambda, r|s^{\pm}|\mu, r') = \frac{1}{N}\sum_{s\in\mathbb{Z}}\langle\lambda, r|p_{2s+1}|\mu, r'\rangle\delta(S_\lambda - S_\mu = \pm 1) \tag{12}$$

where $p_m = \sum_{i=1}^{N} x_i^m$ is the power sum symmetric function. One can identify $s^{\pm}$ with $(s^+ + s^-)/2$ and replace the δ by $1/2\delta(|S_\lambda - S_\mu| = 1)$. In this paper we only study form factors which satisfy this selection rule. By symmetry between the contributions of p_m and p_{-m} (or equivalently the duality relation (see [17]), one may finally only consider the form factors

$$(\lambda, r|s^{\pm}|\mu, r') \equiv \frac{1}{N}\sum_{s\in\mathbb{Z}^+}\langle\lambda|p_{2s+1}|\mu\rangle\delta_{r,r'}\delta(|S_\lambda - S_\mu| = 1). \tag{13}$$

2.3. $\mathfrak{gl}_2$ *Jack technology*

To compute transition matrix elements such as (13), we need some results from the mathematical literature on Macdonald polynomials [33].

We consider a generic partition as in figure 1. It is a tableau made of cases s labelled by their row number i and their column number j. The length (the number of cases) of the row i is λ_i and the length of the column j is λ'_j. One then defines

Definitions for partitions:

Length: $l(\lambda) = \max(\lambda'_j)$;
Cardinal: $|\lambda| = \sum\lambda_i$ and we note $\lambda \vdash |\lambda|$;
Arm-length: $a(s) = \lambda_i - j$;
Leg-length: $l(s) = \lambda'_j - i$;
Arm-colength: $a'(s) = j - 1$;
Leg-colength: $l'(s) = i - 1$;
Content: $c(s) = a'(s) - l'(s)$;
Hook-length: $h(s) = a(s) + l(s) + 1$;
$C(\lambda) = \{s \in \lambda | c(s) \equiv 0 \bmod 2\}$;
$H(\lambda) = \{s \in \lambda | h(s) \equiv 0 \bmod 2\}$.

The quantities of interest in the present work are the norm of a given state, the expansion of power sums on Jack polynomials (thus on physical states), and finally the Pieri formula. The latter gives the development on Jack polynomials of the product of a Jack polynomial and an elementary function $e_r = P_{1^r}^{(\gamma,2)}$. This is the only formula available to compute form factors in the space of symmetric polynomials.

They read

Norm

$$\left\langle P_\lambda^{(\gamma,2)} \middle| P_\lambda^{(\gamma,2)} \right\rangle = \langle 1|1\rangle \prod_{C(\lambda)} \frac{a'(s)+\gamma(N-l'(s))}{a'(s)+1+\gamma(N-l'(s)-1)} \prod_{H(\lambda)} \frac{a(s)+1+\gamma l(s)}{a(s)+\gamma(l(s)+1)}. \tag{14}$$

Expansion of power sums

$$p_{2s+1} = \sum_{\lambda \vdash 2s+1} \chi_\lambda P_\lambda^{(\gamma,2)} \tag{15}$$

$$\chi_\lambda = (-)^{n(\lambda)} \frac{\prod_{C(\lambda)\setminus(1,1)} (a'(s)-\gamma l'(s)}{\prod_{H(\lambda)} a(s)+1+\gamma l(s)} \qquad \text{for} \quad |C(\lambda)| = |H(\lambda)|+1. \tag{16}$$

Pieri formula

$$P_\mu^{(\gamma,2)} e_r = \sum_\lambda \psi'_{\lambda/\mu} P_\lambda^{(\gamma,2)} \tag{17}$$

$$\psi'_{\lambda/\mu} = \prod_{C_{\lambda/\mu}\setminus R_{\lambda/\mu}} \frac{b_\lambda(s)}{b_\mu(s)} \tag{18}$$

$$b_\lambda = \frac{a(s)+\gamma(l(s)+1)}{a(s)+1+\gamma l(s)} \qquad \text{if} \quad s \in H(\lambda) \quad 1 \text{ otherwise} \tag{19}$$

with $\lambda - \mu$ being a vertical r-strip (at maximum 1 box per row, for a total of r), $C_{\lambda/\mu}$ (resp. $R_{\lambda/\mu}$) being the union of columns (resp. rows) that intersect $\lambda - \mu$.

3. Transition matrix elements in the Haldane–Shastry model

We now specify this mathematical background to the case of the Haldane–Shastry spin chain. The $\beta \to \infty$ limit simplifies the algebra to a great extent. The eigenstates described above can be interpreted as multi-spinon states. We give their physical properties in the following. Then a general expression for the matrix elements of the spin operator is presented along with the closed analytical result in the case of few-spinon states.

3.1. Spinon interpretation

In the Haldane–Shatry framework, all physical quantities are rewritten in terms of sums over the columns λ'_j of the partition λ:

$$P_\sigma = \pi r + \pi \sum_j \frac{2w_j}{N} \tag{20}$$

$$E_\sigma = \pi^2 \left[(1-(-)^j/N)\frac{2w_j}{N} - \left(\frac{2w_j}{N}\right)^2 + (1-(-)^r)/2N \right] \tag{21}$$

$$S_\sigma = \sum_j [\lambda'_j - 2w_j] \tag{22}$$

$$w_j = \begin{cases} \lfloor \frac{\lambda'_j}{2} \rfloor & \text{for} \quad j+r \text{ even} \\ \lceil \frac{\lambda'_j}{2} \rceil & \text{for} \quad j+r \text{ odd.} \end{cases} \tag{23}$$

At the thermodynamic limit (which only is of interest), distinctions between even and odd disappear for the momentum and the energy. Defining $x_j = \pi\lambda'_j/N \in [0,\pi]$, they are

$$P_\sigma = \pi r + \sum_j x_j \tag{24}$$

$$E_\sigma = \sum_j x_j(\pi - x_j). \tag{25}$$

One recognizes the dispersion relation of spinons. An excitation can then be described by a zero mode r and a set of particles called spinons defined by each column of a tableau λ. For the sake of simplicity we will write such a state $\{m_j = \lambda'_j\}_r$. r can be discarded in most of the physical applications.

Now we can express all the Jack polynomial technology in terms of the spinons' quantum numbers m_j. We will use the short cut notation

$$\gamma_{\text{even}}(m) = \frac{\Gamma(\lfloor m/2 \rfloor + 1)\Gamma(1/2)}{\Gamma(\lfloor m/2 \rfloor + 1/2)} \simeq \sqrt{\frac{Nx}{2}} \tag{26}$$

$$\gamma_{\text{odd}}(m) = \frac{m}{2\gamma_{\text{even}}(m)} \simeq \frac{1}{\pi}\sqrt{\frac{Nx}{2}}. \tag{27}$$

The norm is

$$N_{\{m_j, j=1,\ldots,n\}} \equiv \frac{\langle P_{\{m_j\}} | P_{\{m_j\}} \rangle}{\langle 1|1\rangle} = \prod_{i=1}^{n} \frac{\gamma_{i+1}(N)}{\gamma_{i+1}(N-m_i)} \prod_{1 \leqslant i \leqslant j \leqslant n} \frac{\gamma_{i-j}(m_i - m_j)}{\gamma_{i-j}(m_i - m_{j+1})} \simeq \prod_i \sqrt{\frac{2\pi}{NE_i}}. \tag{28}$$

One shows that the power sum operators decompose into two-spinon states (m_1, m_2) such that $m_1 + m_2 = 2s + 1$ with

$$\chi_{(m_1,m_2)} = (-)^{m_2}\gamma_0(m_1 - m_2). \tag{29}$$

Specifying the Pieri formula to $\mu \equiv (m_1, \ldots, m_n), \lambda \equiv (p_1, \ldots, p_{n+1})$—with possibly $p_{n+1} = 0$—it gives

$$\psi'_{\lambda/\mu} = \prod_{1 \leqslant i \leqslant j \leqslant n} \frac{\gamma_{i-j}(p_i - p_{j+1})\gamma_{i-j}(m_i - m_j)}{\gamma_{i-j}(p_i - m_j)\gamma_{i-j}(m_i - p_{j+1})}. \tag{30}$$

3.2. Matrix elements

Different strategies apply to the evaluation of the transition matrix elements $\langle\lambda|p_{2s+1}|\mu\rangle$. The first one is to write μ and λ into elementary functions e by inversion of the Pieri formula (17), use the decomposition (15), and apply the Pieri formula successively on the multi-e state. This solution is quite impractical, because inverting the Pieri formula becomes increasingly difficult with the number of spinons considered. It is trivial for one spinon, and leads to a rather cumbersome expression already for two spinons. Still, we can use it to rewrite the power sums as a sum of $2e$ states. The result is interestingly simple

$$p_{2s+1} = \frac{1}{2}\sum_{r=0}^{s}(-)^{s+r+1}(2r+1)e_{s+r+1}e_{s-r}. \tag{31}$$

To prove this, one uses (17) on (31), which leads to (29), thanks to the equality

$$1 = \frac{1}{2}\sum_{r=0}^{s}\frac{1}{\gamma_0(r)\gamma_0(s-r)}. \tag{32}$$

It gives way to a generic strategy: use decomposition (31) and use the Pieri formula twice. The result is (with normalized states)

$$\langle\lambda|p_{2s+1}|\mu\rangle = \frac{1}{2}\sum_{\nu}(-)^{|\lambda|-|\nu|}(|\mu|+|\lambda|-2|\nu|)\Psi'_{\lambda\setminus\nu}\Psi'_{\nu\setminus\mu}\sqrt{\frac{N_\lambda}{N_\mu}} \tag{33}$$

with $|\lambda|-|\mu| = 2s+1$.

We will now evaluate the contribution of several channels. Equation (33) does not give a closed analytic expression for the form factors. Only in a few cases can it be so reduced. The results may be conjectural, then confirmed through the following sum rule,

$$\sum_{m>0}\langle\mu|p_m p_{-m}|\mu\rangle \xrightarrow[N\to\infty]{} |\mu| \tag{34}$$

which can be proved by a simple Jack polynomial algebra.

$0\to 2$ *spinons.* This is the only channel present at zero temperature. It has been conjectured by Haldane, then proved using the symplectic ensemble [16], and finally Yamamato *et al* [18] obtained it using Uglov's technology.

We recall the result and give the thermodynamic limit

$$\langle(m_1,m_2)|p_{m_1+m_2}|0\rangle = \sqrt{\frac{\gamma_0(m_1-m_2)\gamma_1(m_1-m_2)\gamma_0[N]\gamma_1[N]}{\gamma_1(m_1)\gamma_0(N-m_1)\gamma_0(m_2)\gamma_1(N-m_2)}} \simeq \sqrt{\frac{\pi(x_1-x_2)}{\sqrt{E(m_1)E(m_2)}}}. \tag{35}$$

$1\to 1$ *spinon*

$$\langle(m')|p_{m'-m}|(m)\rangle = \sqrt{\frac{\gamma_0(m')\gamma_0(N-m)}{\gamma_0(m)\gamma_0(N-m')}} \simeq \left(\frac{x'(\pi-x)}{x(\pi-x')}\right)^{1/4}. \tag{36}$$

$2\to 2$ *spinons*

$$\langle(m'_1,m'_2)|p_m|(m_1,m_2)\rangle = \sqrt{\frac{\gamma_0(m'_2)\gamma_1(m'_1)\gamma_0(N-m_1)\gamma_1(N-m_2)}{\gamma_0(m_2)\gamma_1(m_1)\gamma_0(N-m'_1)\gamma_1(N-m'_2)}}$$
$$\times\begin{cases}\sqrt{\frac{\gamma_0(k)\gamma_1(k')}{\gamma_0(k')\gamma_1(k)}} & \text{if} \quad m'_1=m_1\\ \sqrt{\frac{\gamma_0(k')\gamma_1(k)}{\gamma_0(k)\gamma_1(k')}} & \text{if} \quad m'_2=m_2\\ \sqrt{\frac{\gamma_0(k')\gamma_1(k)}{\gamma_0(k)\gamma_1(k')}}G(m,l=m'_2-m_2,k,k') & \text{otherwise.}\end{cases} \tag{37}$$

$$G(m,l,k,k') = \sum_{i=1}^{l} (-)^{i(k+l+1)} \frac{\gamma_1(i)}{\gamma_0(i)} \frac{\Gamma\left(\left\lceil \frac{m-l+i}{2} \right\rceil\right)}{\Gamma\left(\left\lfloor \frac{m-l}{2} \right\rfloor + 1\right)} \frac{\Gamma\left(\left\lceil \frac{l}{2} \right\rceil\right)}{\Gamma\left(\left\lfloor \frac{l-i}{2} \right\rfloor + 1\right)} \frac{\Gamma\left(\left\lfloor \frac{k-i}{2} \right\rfloor + \frac{1}{2}\right)}{\Gamma\left(\left\lfloor \frac{k}{2} \right\rfloor + \frac{1}{2}\right)} \frac{\Gamma\left(\left\lceil \frac{k'}{2} \right\rceil + \frac{1}{2}\right)}{\Gamma\left(\left\lceil \frac{k'+i}{2} \right\rceil + \frac{1}{2}\right)} \tag{38}$$

with $m = m'_1 + m'_2 - m_1 - m_2$, $k = m_1 - m_2$, $k' = m'_1 - m'_2$. This can be interpreted as an $SU(2)$ generalization of result (28) of [30] for $g = 1/2$ quasi-particles.

Higher channels. It is generally not possible to obtain a closed analytical expression for the other channels, except in a few cases. Such are, for example, $[(m_1) \to (m_1, n, n')]$ and $[(m_1) \to (n, n', m_1)]$, which equal $[0 \to (n, n')]$. Nevertheless, formula (33) can be used to give exact numerical results. Getting the thermodynamic limit directly is a challenging but fruitful work. We leave it as an open question.

4. Correlation functions at finite temperature

Before addressing the computation of the DSSF, general considerations on finite-temperature correlation functions are needed. For a local operator $\mathcal{O}$, it is

$$\langle \mathcal{O} \rangle_T = \frac{\sum_n \sum_{\lambda_n} \langle \lambda_n | \mathcal{O} | \lambda_n \rangle \exp(-\beta E_{\lambda_n})}{\sum_n \sum_{\lambda_n} \exp(-\beta E_{\lambda_n})} \tag{39}$$

where n is the number of quasi-particles, and λ_n is a state with n quasi-particles.

As recalled in the introduction, divergences appear in the correlators of the right-hand side, which need to be resummed. For ITF, LeClair and Mussardo proposed such a resummation as a form factor expansion on the basis of the asymptotic particle states in the zero-temperature theory [23],

$$\langle \mathcal{O} \rangle_T = \sum_n \sum_{\lambda_n} \langle \lambda_n | \mathcal{O} | \lambda_n \rangle_{\text{irr}} \prod_{i=1}^{n} \bar{n}_T(E_{\lambda_i}). \tag{40}$$

The irreducible form factor $\langle \lambda_n | \mathcal{O} | \lambda_n \rangle_{\text{irr}}$ is obtained, thanks to the form factor bootstrap (FFB), and $\bar{n}_T(E)$ is the filling factor determined by the thermodynamic Bethe ansatz (TBA).

In the following, we show that the FFE apply also for the Haldane–Shastry spin chain in the thermodynamic limit. First we obtain the thermodynamic properties of the spinons, then give the expression of the irreducible form factor.

4.1. Exclusion statistics

A form factor expansion similar to (40) was proposed for CFTs [29]. It led to the identification of a two-body $\boldsymbol{S}$-matrix for the CFT in the thermodynamic limit,

$$\boldsymbol{S} = \exp[2\mathrm{i}\pi(\boldsymbol{\delta} - \boldsymbol{K})\Theta(\theta)] \tag{41}$$

where $\boldsymbol{K}$ is the exclusion statistics matrix of the quasi-particles of the theory (which play the role of asymptotic states).

Fractional exclusion statistics is a tool introduced by Haldane [15] for the analysis of strongly correlated many-body systems. It is only based on the assumption that the Hilbert space is finite dimensional and extensive, i.e. particles are excitations of the considered condensed matter system, so it is a very generic concept. The statistics are encoded in a matrix $\boldsymbol{K} = (K_{ij})$ correponding to the reduction of the available Hilbert space for particle of type i by filling a one-particle state by a particle of type j. This is then a generalization of the Pauli principle.

For spin-1/2 spinons with species $i = \pm$, the statistical matrix is [34]

$$\boldsymbol{K} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \tag{42}$$

As such, their one-particle distribution functions generalize the familiar Fermi–Dirac and Bose–Einstein ones. They are derived from one-particle grand canonical partition functions G_i given by the IOW equations [35–37]

$$\left(\frac{G_i - 1}{G_i}\right) \prod_j G_j^{\boldsymbol{K}_{ij}} = z_i \tag{43}$$

where G_i depends on the generalized fugacities $z_j = \mathrm{e}^{\beta(\mu_j - \varepsilon)}$. The one-particule distribution functions are obtained through

$$n_i(\varepsilon) = z_i \frac{\partial}{\partial z_i} \log \prod_j G_j. \tag{44}$$

In our case, where species are $i = (+, -)$, we obtain

$$n_\pm(\varepsilon) = \frac{z_\pm}{\sqrt{1 + \left(\frac{z_+ - z_-}{2}\right)^2}} \frac{\pm\left(\frac{z_+ - z_-}{2}\right) + \sqrt{1 + \left(\frac{z_+ - z_-}{2}\right)^2}}{\left(\frac{z_+ + z_-}{2}\right) + \sqrt{1 + \left(\frac{z_+ - z_-}{2}\right)^2}} \xrightarrow{\mu^+ = \mu^-} \frac{1}{\exp(\beta E) + 1} \tag{45}$$

from which one deduces that at zero magnetic field, these distributions match the Fermi–Dirac distribution function. It means that spinons can be considered as having fermionic statistics. This is indeed what is done when they are labelled by ordered numbers within partitions; the spin degree of freedom is hidden. The norm formula (28) shows that for a multi-spinon state, the labels have to be strictly decreasing, as for spinless fermions.

4.2. Irreducible form factors

For ITFs, the definition of the irreducible form factors comes *a priori* from the FFB. Within our framework (based on [29]) it is not even necessary. Their definition is

$$\langle \lambda_n | \mathcal{O} | \lambda_n \rangle = \langle \lambda_n | \mathcal{O} | \lambda_n \rangle_{\mathrm{irr}} + \sum_{\overline{\lambda_n}} \langle \overline{\lambda_n} | \mathcal{O} | \overline{\lambda_n} \rangle_{\mathrm{irr}} \tag{46}$$

with $\overline{\lambda_n}$ being a substate of λ_n (a substate being a state where some of the spinons have been taken out). Nonetheless, the FFB ensures that irreducible form factors do not carry divergences, which we cannot prove here. This calls for further understanding of form factors in fractional statistics theories.

Here follows a sketch of the proof that the irreducible form factors (46) give the correct FFE (40). It only uses the fact that the colourless spinons are free fermions, which shows up by the strict ordering of their quantum numbers within a multi-spinon state. As such, the proof is similar to [38], but in the discrete case.

We first remark that $\langle 0 | \mathcal{O} | 0 \rangle_{\mathrm{irr}}$ trivially comes with the factor 1. Let us consider the factor of $D(m) = \langle (m) | \mathcal{O} | (m) \rangle_{\mathrm{irr}}$. The contribution C_n from n quasi-particles is obtained recursively, isolating the spinons whose labels match with m:

$$C_n = \frac{1}{Z} \sum_{m_1>\cdots>m_n} \sum_i D(m_i) \prod_{i=1}^{n} \exp(-\beta E_{m_i}) \tag{47}$$

$$= \frac{1}{Z} \sum_{m_1>\cdots>m_{n-1}} \prod_{i=1}^{n-1} \mathrm{e}^{-\beta E_{m_i}} \sum_m D(m)\, \mathrm{e}^{-\beta E_m} \left(1 - \sum_{i=1}^{n-1} \delta_{m,m_i}\right) \tag{48}$$

$$= \frac{1}{Z} \left[\sum_{m_1>\cdots>m_{n-1}} \prod_{i=1}^{n-1} \mathrm{e}^{-\beta E_{m_i}} \sum_m D(m)\, \mathrm{e}^{-\beta E_m} - \sum_{m_1>\cdots>m_{n-2}} \prod_{i=1}^{n-2} \mathrm{e}^{-\beta E_{m_i}} \sum_m D(m)\, \mathrm{e}^{-2\beta E_m} \left(1 - \sum_{i=1}^{n-2} \delta_{m,m_i}\right) \right] \tag{49}$$

$$= \cdots = \frac{\sum_m D(m)\, \mathrm{e}^{-\beta E_m} \sum_{i=0}^{n-1} (-)^i z_{n-1-i}\, \mathrm{e}^{-i\beta E_m}}{\sum_{i=0}^{\infty} z_i} \tag{50}$$

$$z_i = \sum_{m_1>\cdots>m_i} \prod_{j=1}^{i} \exp(-\beta E_{m_j}). \tag{51}$$

Then summing over n gives

$$\sum_{n=1}^{\infty} C_n = \sum_m D(m)\, \mathrm{e}^{-\beta E_m} \frac{\sum_{n=1}^{\infty} \sum_{i=0}^{n-1} (-)^i z_{n-1-i}\, \mathrm{e}^{-\mathrm{i}\beta E_m}}{\sum_{i=0}^{\infty} z_i} \tag{52}$$

$$= \sum_m D(m)\, \mathrm{e}^{-\beta E_m} \frac{\sum_{i=0}^{\infty} (-)^i\, \mathrm{e}^{-\mathrm{i}\beta E_m} \sum_{n=i+1}^{\infty} z_{n-1-i}}{\sum_{i=0}^{\infty} z_i} \tag{53}$$

$$= \sum_m D(m)\, \mathrm{e}^{-\beta E_m} \sum_{i=0}^{\infty} (-)^i\, \mathrm{e}^{-\mathrm{i}\beta E_m} \tag{54}$$

$$= \sum_m D(m) \frac{1}{\exp(\beta E_m) + 1}. \tag{55}$$

The demonstration follows the same lines for the higher form factors. Equation (40) is quite convenient. Indeed, the denominator is suppressed, so it can be seen as a perturbative series, provided that it is converging. It also has a clear physical interpretation with quasi-particles and filling factors. Now it can be applied to the particular case of the dynamical spin structure factor (DSSF).

5. Finite-temperature spin structure factor

Now having all the methodological ingredients we address the main subjet of our paper: the computation of the finite-temperature spin structure factor of the Haldane–Shastry spin chain. We first recall the zero-temperature result. It was first obtained by Haldane and Zirnbauer [16] using the supermatrix method. It can also be accessed with $\mathfrak{gl}_2$ Jack polynomials [18, 19]. Only the $[0 \to 2]$ channel contributes, so it is straightforwardly

$$S_0(q,\omega) = \int_0^{\pi} \mathrm{d}x_1 \int_0^{\pi} \mathrm{d}x_2 \frac{\pi |x_1 - x_2|}{\sqrt{E(x_1)E(x_2)}} \delta(x_1 + x_2 - q)\delta(E(x_1) + E(x_2) - \omega). \tag{56}$$

At finite temperature, all channels contribute; one needs to sum them all through a FFE. We detail in the following the static structure factor $S_T(q) = \int \mathrm{d}\omega\, S_T(q,\omega)$ and the DSSF.

5.1. Static structure factor

The static spin–spin correlator can be described by a one-point function

$$S_T(q = 2\pi s/N) = \left\langle \frac{1}{N} p_{-(2s+1)} p_{(2s+1)} \right\rangle_T \tag{57}$$

(this is not the exact one, but equal at the thermodynamic limit). To compute it, we use the one-point FFE and the transition matrix elements obtained before.

Due to the complexity of the form factor and the lack of FFB, it is hardly impossible to obtain analytical results on the correlation functions. The only point for which we could obtain the correlator exactly is $q = 0$. We compute it as the thermodynamic limit ($N \to \infty$) of (57) at $s = 0$. The FFE is

$$\langle p_{-1} p_1 \rangle_T = \frac{1}{N} \sum_{\mu} n_F(E_\mu) \langle \mu | p_{-1} p_1 | \mu \rangle_{\text{irr}} \tag{58}$$

$$= \sum_{n=1}^{\infty} F_n \tag{59}$$

$$F_n = \sum_{m_1 > \cdots > m_n} \prod_{i=1}^{n} n_F(E_{m_i}) \langle \{m_1, \ldots, m_n\} | p_{-1} p_1 | \{m_1, \ldots, m_n\} \rangle_{\text{irr}}. \tag{60}$$

In the appendix we show that

$$F_n = \int_0^{\pi} \mathrm{d}x \, [n_F(E(x))]^n \tag{61}$$

from which we conclude

$$S_T(q = 0) = \sum_{n=1}^{\infty} \int_0^{\pi} \mathrm{d}x \left(\frac{1}{\exp(\beta E(x)) + 1} \right)^n \tag{62}$$

$$= \int_0^{\pi} \mathrm{d}x \, \mathrm{e}^{-\beta E(x)}. \tag{63}$$

This result brings strong evidence of the power of the FFE to obtain finite-temperature correlation functions. This is the main achievement of the paper. Let us remark here that F_n is the contribution of the n-spinon states in the FFE. They appear to be of the same order in temperature, meaning that the FFE is not a low-temperature expansion[2]. It really is a perturbative expansion, in the sense that the different contributions decrease exponentially with the number of spinons involved.

Nonetheless, $S_T(q = 0)$ is a low-energy feature of the theory that can be obtained with a bosonization approach. As in [30], we observe that the FFE is mostly powerful in regimes where other simpler methods apply. To calculate the whole static structure factor, we then rely on numerics. We work in the infinite-temperature regime to compare the FFE with the expected result. In this limit, correlations are only local and we expect the static structure factor not to depend on the momentum and have the value $S_{T=\infty}(q = 0) = 1$.

To observe the FFE perturbative power, we studied the contribution from 2- and (2 + 4)-multi-spinon states. What we understand here as two-spinons is the sum of the contributions of the $[0 \leftrightarrow 2]$ and $[1 \leftrightarrow 1]$ channels, and as four-spinons the $[1 \leftrightarrow 3]$ and $[2 \leftrightarrow 2]$ channels.

[2] This would have been expected on the grounds that more thermal energy is needed to excite more particles. This is no longer true for massless particles.

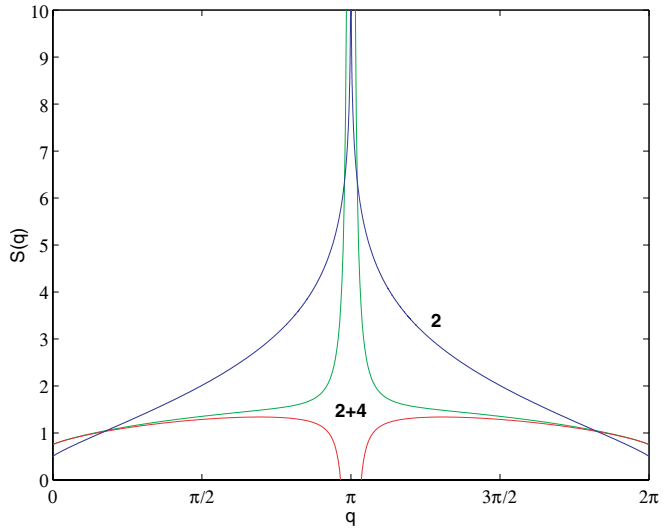

Figure 2. Static spin structure factor. **2** is the contribution from two-particle form factors with $N = 10\,000$; **2+4** is the contribution from two- and four-particle form factors with $N = 500$; the two different curves are obtained with two labellings of the spinons: for the upper curve, $0 < m < N$ (the normal one), for the lower curve, $0 \leqslant m \leqslant N$; they should match in the $N \to \infty$ limit (up to a δ peak on $q = \pi$), so we expect the thermodynamic limit to lie in between the two curves.

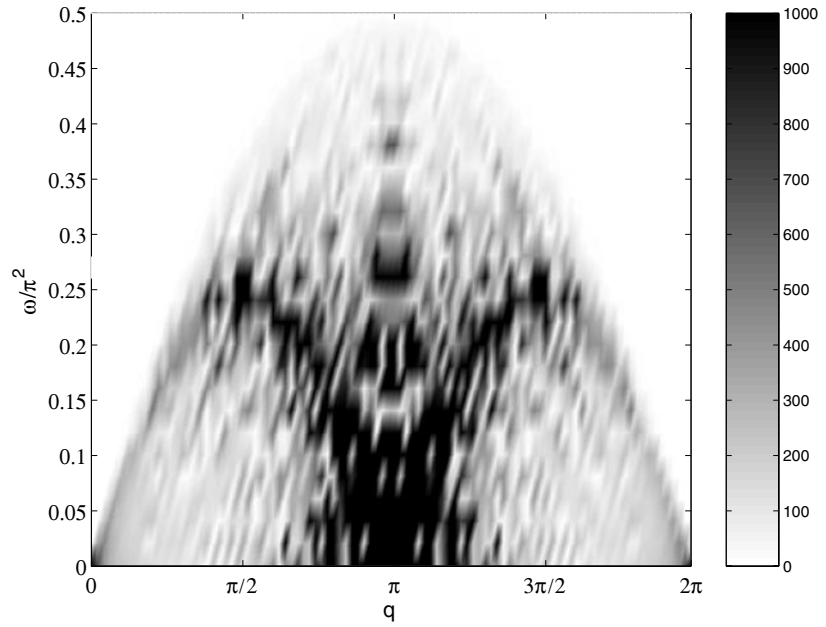

Figure 3. Dynamic spin structure factor: the contributions from two- and four-particle form factors with $N = 500$, in absolute value.

Results are gathered in figure 2. At first we observe the convergence to be rather good. This strongly supports our approach. Still problems arise in the vicinity of $q = \pi$. Finite-size

effects starts appearing at the level of four-spinon contributions, and increase with higher number of spinons. The only way to tackle them is to have an enormous computing time. We admit this is the most important weakness of the approach. This emphasizes the need for a direct thermodynamic limit computation.

5.2. Dynamic structure factor

We show in figure 3 the sum of the contributions from two- and four-spinons to the infinite-temperature DSSF. As in the previous paragraph, finite-size effects clearly appear around $q = \pi$. This makes the comparison to experiments hardly possible. We still note that some basic features, such as the double arch shape, agree qualitatively. This feature, already present at zero temperature, comes from form factors where the spin operator acts on only one of the spinons in a multi-spinon excitation, leading to the spinon dispersion relation. We leave more comments to the concluding section.

6. Conclusion

In this paper we studied the finite-temperature spin–spin correlation function of the Haldane–Shastry spin chain. We used the multi-spinon basis obtained as the infinite coupling limit of the Yangian basis of the spin Calogero–Sutherland model. Form factors of the spin operators were computed, thanks to the $\mathfrak{gl}_2$ Jack polynomial technology. These form factors were gathered within a form factor expansion of finite-temperature correlation functions to give physical quantities directly comparable with the experiments. Though the Haldane–Shastry is not a quantitatively appropriate model to describe real spin chains such as $KCuF_3$, it should be at the qualitative level. As any finite-temperature properties for the Heisenberg model are as yet beyond reach, serious theoretical insights are brought by the Haldane–Shastry model. We recall that it is simpler due to the $1/r^2$ interaction responsible for the thermodynamic freedom of the spinons.

Our main analytical results are: the formal expression of any form factor of the theory, a closed analytical expression for the simplest of them, the proof of the finite-temperature FFE and the value at zero momentum of the static structure factor. Along with numerical results, they provide evidence of the FFE technique to approximate finite-temperature correlations. As a drawback, it demands huge computation power. This calls for further theoretical refinement.

Work is in progress to provide the Haldane–Shastry model with integrability features such as the FFB. Within such a description, thermodynamic irreducible form factors would be obtained much more easily using scattering properties. Problems to developing this approach come from the intrinsic discrete nature of the Haldane–Shastry spinons. But the necessary effort has to be made to gather comprehensively ITFs and CFTs into a common framework. Further understanding is also needed in the link between the Haldane–Shastry model and the symplectic random matrix ensemble used in [16]. The analysis performed at zero temperature could be extended to finite temperature. It would also be interesting to treat the finite-temperature dynamics of other inverse-square interaction models and compare to their zero-temperature exact result [20, 22]. It comes as a simple generalization of the method used in the present paper.

At the numerical level, finite size calculations can be performed. For sufficiently small sizes, the exact correlation functions are accessible using the analytical results of this paper. The FFE cannot be used for it is based on the thermodynamic limit, instead a direct application of (39) is necessary. Using the Yangian multiplets directly is another solution, as in [39] for zero temperature.

Acknowledgments

The author thanks K Schoutens and J-S Caux for fruitful discussions.

Appendix. Zero-momentum limit of the static structure factor

In this section we show that

$$F_n = \sum_{m_1>\cdots>m_n} \prod_{i=1}^{n} n_F\big(E_{m_i}\big)\langle\{m_1,\ldots,m_n\}|p_{-1}p_1|\{m_1,\ldots,m_n\}\rangle_{\rm irr}$$
$$= \int_0^\pi \mathrm{d}x\,[n_F(E(x))]^n. \tag{A.1}$$

It is clear from the conservation laws that the intermediate states for each form factor are only different from the initial state by an increase of 1 of one of the spinons' m. Quite generally, we thus need

$$|\langle\{m_1,\ldots,m_k+1,\ldots,m_n\}|p_1|m_1,\ldots,m_k,\ldots,m_n\}\rangle|^2 \simeq \prod_{i\neq k} \alpha^{\mathrm{sgn}(i-k)}_{(-)^{i-k}}(m_k-m_i) \tag{A.2}$$

$$\alpha^{+}_{\pm}(m) = \left(\frac{\gamma_0(m)\gamma_1(m+1)}{\gamma_1(m)\gamma_0(m+1)}\right)^{\pm} \tag{A.3}$$

$$\alpha^{-}_{\pm}(-m) = \left(\frac{\gamma_1(m)\gamma_0(m-1)}{\gamma_0(m)\gamma_1(m-1)}\right)^{\pm}. \tag{A.4}$$

Explicitly developing the irreducible form factors and separating different affected m, we obtain

$$F_n = \sum_{m_1>\cdots>m_n} \prod_{i=1}^{n} n_F\big(E_{m_i}\big) \sum_k F_n^k \tag{A.5}$$

$$F_n^k = \sum_{J\subset\{1,\ldots,k-1,k+1,\ldots,n\}} (-)^{n-1-|J|} \prod_{i\in J} \alpha^{\mathrm{sgn}(i-k)}_{(-)^{d_J(k,i)}}(m_k-m_i) \tag{A.6}$$

$$= \sum_{J^>\subset\{k+1,\ldots,n\}} (-)^{n-k-|J^>|} \prod_{i\in J^>} \alpha^{+}_{(-)^{d^>_J(k,i)}}(m_k-m_i)$$
$$\times \sum_{J^<\subset\{1,\ldots,k-1\}} (-)^{k-1-|J^<|} \prod_{i\in J^<} \alpha^{-}_{(-)^{d^<_J(k,i)}}(m_i-m_k) \tag{A.7}$$

with $d_J(k,i)$ the distance between k and i in the subset J. Thus, we can write $F_n^k = F_n^{k>}F_n^{k<}$.

We will now obtain $\sum_{\{m_{i>k}\}} F_n^{k>}$ in a recursive way (the proof for $F_n^{k>}$ follows the same lines), putting $k=1$ without loss of generality. We first perform the sum over m_n. Writing $I=\{2,\ldots,n-1\}$, one separates the ensemble of J as

$$\{J\subset I\cup\{n\}\} = \{J+J\cup\{n\},\, J\subset I\}$$
$$= \{J+J\cup\{n\},\, J\in I,\, |J|\,\text{even}\}\cup\{J+J\cup\{n\},\, J\subset I,\, |J|\,\text{odd}\}.$$

In the first (resp. second) subset, the distance in J between 1 and n is odd (resp. even). This proves that

$$F_n^{1>} = (\alpha^{+}_{-}(m_1-m_n)-1)F_{n-1}^{1>\mathrm{even}} + \big(\alpha^{+}_{+}(m_1-m_n)-1\big)F_{n-1}^{1>\mathrm{odd}} \tag{A.8}$$

where the superscripts 'even' and 'odd' correspond to restrictions of the expression of $F_{n-1}^{1>}$ to J subsets with even or odd cardinal. Repeating this recursion one ends up with

$$F_n^{1>} = F_{n-2}^{1>\text{even}}[(\alpha_+(m_1 - m_n) - 1)\alpha_-^+(m_1 - m_{n-1}) + 1 - \alpha_-^+(m_1 - m_n)] + F_{n-2}^{1>\text{odd}}\big[(\alpha_-^+(m_1 - m_n) - 1)\alpha_+^+(m_1 - m_{n-1}) + 1 - \alpha_+^+(m_1 - m_n)\big]. \quad (A.9)$$

Then one easily shows that

$$\big[\big(\alpha_+^+(m_1 - m_n) - 1\big)\alpha_-^+(m_1 - m_{n-1}) + 1 - \alpha_-^+(m_1 - m_n)\big] \simeq \alpha_-^+(m_1 - m_n)\delta_{m_n - m_{n-1}} \quad (A.10)$$

$$\big[(\alpha_-^+(m_1 - m_n) - 1)\alpha_+^+(m_1 - m_{n-1}) + 1 - \alpha_+^+(m_1 - m_n)\big] \simeq \alpha_+^+(m_1 - m_n)\delta_{m_n - m_{n-1}} \quad (A.11)$$

thus proving the recursion

$$F_n^{1>} = F_{n-1}^{1>}\delta_{m_n - m_{n-1}}. \quad (A.12)$$

As this is true only for $n > 1$, we obtain for F_n^k the following:

$$F_n^1 = (\alpha_-^+(m_1 - m_2) - 1)\delta_{m_2,\ldots,m_n} \quad (A.13)$$

$$F_n^{k\neq 1,n} = (\alpha_-^+(m_k - m_{k+1}) - 1)(\alpha_+^-(m_k - m_{k-1}) - 1)\delta_{m_1,\ldots,m_{k-1}}\delta_{m_{k+1},\ldots,m_n} \quad (A.14)$$

$$F_n^n = (\alpha_+^-(m_n m_{n-1}) - 1)\delta_{m_1,\ldots,m_{n-1}}. \quad (A.15)$$

Now, one has $\alpha_-^+(m_k - m_{k+1}) - 1 \equiv \delta_{m_k,m_{k+1}}$ and $\alpha_+^-(m_k - m_{k-1}) - 1 \equiv 0$ so that

$$F_n^k = \delta_{k,1}\delta_{m_1,\ldots,m_n}. \quad (A.16)$$

Finally

$$F_n = \sum_{m=1}^{N-1}[n_F(E(m))]^n = \int_0^\pi \mathrm{d}x\,[n_F(E(x))]^n \quad (A.17)$$

which completes our proof.

References

[1] Tennant D, Lake B, Nagler S and Frost C unpublished
[2] Heisenberg W 1928 *Z. Phys.* **49** 619
[3] Bethe H 1931 *Z. Phys.* **71** 205
[4] Müller G, Thomas H, Beck H and Bonner J C 1981 *Phys. Rev.* B **24** 1429
[5] Schultz H J 1986 *Phys. Rev.* B **34** 6372
[6] Jimbo M and Miwa T 1995 *Albegraic Analysis of Solvable Lattice Models* (Providence, RI: American Mathematical Society)
[7] Korepin V E, Bogoliubov N M and Izergin A G 1993 *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge: Cambridge University Press)
[8] Bougourzi A, Couture M and Kacir M 1996 *Phys. Rev.* B **54** R12669
[9] Karbach M, Müller G, Bougourzi A, Fledderjohann A and Mütter K-H 1997 *Phys. Rev.* B **55** 12510
[10] Bougourzi A, Karbach M and Müller G 1998 *Phys. Rev.* B **57** 11429
[11] Kitanine N, Maillet J-M, Slavnov N and Terras V 2002 *Nucl. Phys.* B **641** 487
[12] Haldane F 1988 *Phys. Rev. Lett.* **60** 635
[13] Shastry B 1988 *Phys. Rev. Lett.* **60** 639
[14] Haldane F 1988 *Phys. Rev. Lett.* **66** 1529
[15] Haldane F 1991 *Phys. Rev. Lett.* **67** 937
[16] Haldane F and Zirnbauer M 1993 *Phys. Rev. Lett.* **71** 4055
[17] Uglov D 1998 *Commun. Math. Phys.* **191** 663
[18] Yamamoto T, Saiga Y, Arikawa M and Kuramoto Y 2000 *J. Phys. Soc. Japan* **69** 900
[19] Yamamoto T, Saiga Y, Arikawa M and Kuramoto Y 2000 *Phys. Rev. Lett.* **84** 1308
[20] Yamamoto T and Arikawa M 1999 *J. Phys. A: Math. Gen.* **32** 3341

[21] Arikawa M, Yamamoto T, Saiga Y and Kuramoto Y 1999 *J. Phys. Soc. Japan* **68** 3782
[22] Arikawa M, Saiga Y and Kuramoto Y 2001 *Phys. Rev. Lett.* **86** 3096
[23] LeClair A and Mussardo G 1999 *Nucl. Phys.* B **552** 624
[24] Delfino G 2001 *J. Phys. A: Math. Gen.* **34** L161
[25] Mussardo G 2001 *J. Phys. A: Math. Gen.* **34** 7399
[26] Konik R 2001 *Preprint* cond-mat/0105284
[27] Saleur H 2000 *Nucl. Phys.* B **567** 602
[28] Castro-Alvaredo O and Fring A 2002 *Nucl. Phys.* B **636** 611
[29] van Elburg R and Schoutens K 2000 *J. Phys. A: Math. Gen.* **33** 7987
[30] Peysson S and Schoutens K 2002 *J. Phys. A: Math. Gen.* **35** 6471
[31] Polychronakos A 1993 *Phys. Rev. Lett.* **70** 2329
[32] Ha Z and Haldane F 1992 *Phys. Rev.* B **46** 9359
[33] Macdonald I 1995 *Symmetric Functions and Hall Polynomials* (Oxford: Oxford University Press)
[34] Bouwknegt P and Schoutens K 1999 *Nucl. Phys.* B **547** 501
[35] Isakov S 1994 *Mod. Phys. Lett.* B **8** 319
[36] Dasnières de Veigy A and Ouvry S 1994 *Phys. Rev. Lett.* **72** 600
[37] Wu Y 1994 *Phys. Rev. Lett.* **73** 922
[38] LeClair A, Lesage F, Sachdev S and Saleur H 1996 *Nucl. Phys.* B **482** 579
[39] Talstra J and Haldane F 1994 *Phys. Rev.* B **50** 6889